

# Locally finite quasivarieties of MV-algebras.\*

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## Abstract

In this paper we show that every locally finite quasivariety of MV-algebras is finitely generated and finitely based. To see this result we study critical MV-algebras. We also give axiomatizations of some of these quasivarieties.

**Keywords:** MV-algebras, critical algebras, quasivarieties, locally finite quasivarieties

## Introduction

In [4, 5] C.C.Chang introduced MV-algebras in order to give an algebraic counterpart of the Łukasiewicz's many valued propositional calculus. In fact, the class of all MV-algebras, in a termwise equivalent presentation named Wajsberg algebras, is the equivalent variety semantics, in the sense of [2], of this calculus (see [17]).

From the equivalence between the class of MV-algebras and Łukasiewicz logic, it is easy to see that finitary extensions of Łukasiewicz's propositional calculus correspond to subquasivarieties of MV-algebras, and axioms and rules of the calculus correspond with equations and quasiequations, respectively. Hence, finite axiomatizable finitary extensions of Łukasiewicz's propositional calculus correspond with finite axiomatizable quasivarieties of MV-algebras.

In this paper, we study finite axiomatizability of locally finite quasivarieties of MV-algebras. To be precise, we show in Section 2 that locally finite quasivarieties and finitely generated quasivarieties of MV-algebras coincide (Theorem 2.7) and that they are finitely axiomatizable (Theorem 2.10). To prove these results, we give a characterization of critical MV-algebras (Theorem 2.5) and we see that any locally finite quasivariety of MV-algebras is generated by critical MV-algebras (Theorem 2.3).

Finally, in Section 3, we give two examples of locally finite quasivarieties with their finite axiomatizations.

We include a preliminary section, Section 1, containing basic definitions, results and notation used in the paper.

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## 1 Definitions and first properties.

An **MV-algebra** is an algebra  $\mathbf{A} = \langle A, \oplus, \neg, 0 \rangle$  of type  $(2, 1, 0)$  satisfying the following equations:

$$\mathbf{MV1.} \quad (x \oplus y) \oplus z \approx x \oplus (y \oplus z)$$

$$\mathbf{MV2.} \quad x \oplus y \approx y \oplus x$$

$$\mathbf{MV3.} \quad x \oplus 0 \approx x$$

$$\mathbf{MV4.} \quad \neg(\neg x) \approx x$$

$$\mathbf{MV5.} \quad x \oplus \neg 0 \approx \neg 0$$

$$\mathbf{MV6.} \quad \neg(\neg x \oplus y) \oplus y \approx \neg(x \oplus \neg y) \oplus x$$

By taking  $y = \neg 0$  in MV6, we deduce:

$$\mathbf{MV7.} \quad x \oplus \neg x \approx \neg 0.$$

More precisely, if we set  $1 = \neg 0$  and  $x \odot y = \neg(\neg x \oplus \neg y)$ , then  $\langle A, \oplus, \odot, \neg, 0, 1 \rangle$  satisfies all axioms given in [14, Lemma 2.6], and hence the above definition of MV-algebras is equivalent to Chang's definition [4]. We denote the class of all MV-algebras by  $\mathbb{W}$ . Since it is an equational class,  $\mathbb{W}$  is a variety.

Given a *lattice ordered abelian group*  $\mathbf{G} = \langle G, \wedge, \vee, +, -, 0 \rangle$  and  $u \in G \quad u > 0$ , we define the algebra  $\Gamma(\mathbf{G}, u) = \langle [0, u], \oplus, \neg, 0 \rangle$  where

- $[0, u] = \{a \in G : 0 \leq a \leq u\}$ ,
- $a \oplus b = u \wedge (a + b)$ ,  $\neg a = u - a$  and  $0 = 0^{\mathbf{G}}$ .

Then  $\Gamma(\mathbf{G}, u)$  is an MV-algebra. In fact any MV-algebra is isomorphic to the unit segment of some lattice ordered abelian group. Concretely, the category of MV-algebras is equivalent to the category of lattice ordered abelian groups with strong unit (See [6],[14]).

The following MV-algebras play an important role in the paper.

- $[0, 1] = \Gamma(\mathbf{R}, 1)$ , where  $\mathbf{R}$  is the totally ordered group of the reals.
- $[0, 1] \cap \mathbf{Q} = \Gamma(\mathbf{Q}, 1) = \langle \{\frac{k}{m} : k \leq m < \omega\}, \oplus, \neg, 0 \rangle$ , where  $\mathbf{Q}$  is the totally ordered abelian group of the rationals and  $\omega$  represents the set of all natural numbers.

For every  $0 < n < \omega$

- $\mathbf{L}_n = \Gamma(\mathbf{Q}_n, 1) = \langle \{\frac{k}{n} : 0 \leq k \leq n\}, \oplus, \neg, 0 \rangle$ , where  $\mathbf{Q}_n = \{\frac{k}{n} : k \in \mathbb{Z}\}$  is a subgroup of  $\mathbf{Q}$  and  $\mathbb{Z}$  is the set of all integers.

- $L_n^\omega = \Gamma(\mathbf{Q}_n \otimes \mathbf{Z}, (1, 0)) = \langle \{(\frac{k}{n}, i) : (0, 0) \leq (\frac{k}{n}, i) \leq (1, 0)\}, \oplus, \neg, 0 \rangle$ , where  $\mathbf{Z}$  is the totally ordered group of the integers and  $\mathbf{Q}_n \otimes \mathbf{Z}$  is the lexicographic product of  $\mathbf{Q}_n$  and  $\mathbf{Z}$ .

The following theorem states some well-known results on simple and/or finite MV-algebras. (See for instance [6]).

**Theorem 1.1**

1. Every simple MV-algebra is isomorphic to a subalgebra of  $[0, 1]$ .
2. Every finite simple MV-algebra is isomorphic to  $L_n$  for some  $n \in \omega$ .
3. Every finite MV-algebra is isomorphic to a direct product of finite simple MV-algebras.
4.  $L_n \subseteq L_m$  if and only if  $n|m$ . □

We denote by  $\mathbb{I}$ ,  $\mathbb{H}$ ,  $\mathbb{S}$ ,  $\mathbb{P}$ ,  $\mathbb{P}_R$  and  $\mathbb{P}_U$  the operators *isomorphic image*, *homomorphic image*, *substructure*, *direct product*, *reduced product* and *ultraproduct* respectively. We recall that a class  $\mathbb{K}$  of algebras is a **variety** if and only if it is closed under  $\mathbb{H}$ ,  $\mathbb{S}$  and  $\mathbb{P}$ . And a class  $\mathbb{K}$  of algebras is a **quasivariety** if and only if it is closed under  $\mathbb{I}$ ,  $\mathbb{S}$  and  $\mathbb{P}_R$ , or equivalently, under  $\mathbb{I}$ ,  $\mathbb{S}$ ,  $\mathbb{P}$  and  $\mathbb{P}_U$ . Given a class  $\mathbb{K}$  of algebras, the variety generated by  $\mathbb{K}$ , denoted by  $\mathbb{V}(\mathbb{K})$ , is the least variety containing  $\mathbb{K}$ . Similarly, the quasivariety generated by a class  $\mathbb{K}$ , which we denote by  $\mathbb{Q}(\mathbb{K})$ , is the least quasivariety containing  $\mathbb{K}$ . We also recall that a class  $\mathbb{K}$  of algebras is a variety if and only if it is an equational class, and  $\mathbb{K}$  is a quasivariety if and only if it is a quasiequational class.

The subvarieties of  $\mathbb{W}$  are well known:

**Theorem 1.2** [13, Theorem 4.11]  *$\mathbb{K}$  is a proper subvariety of  $\mathbb{W}$  if and only if there exist two disjoint finite subsets  $I, J$  of natural numbers such that*

$$\mathbb{K} = \mathbb{V}(L_i \mid i \in I, L_j^\omega \mid j \in J). \quad \square$$

An algebra  $\mathbf{A}$  is **locally finite** if and only if every finitely generated subalgebra is finite. A class  $\mathbb{K}$  is **locally finite** if and only if every member of  $\mathbb{K}$  is locally finite.

If  $\mathbb{V}$  is a variety, then  $\mathbb{V}$  is locally finite if and only if all free algebras with respect  $\mathbb{V}$  over a finite set of generators are finite [3, page 69].

If  $\mathbb{K}$  is a quasivariety, then  $\mathbb{K}$  is locally finite if and only if  $\mathbb{K}$  is contained in a locally finite variety.

A variety, or a quasivariety, is **finitely generated** if it is generated by a finite set of finite algebras.

**Theorem 1.3** [3, page 70] *Every finitely generated variety is locally finite.*  $\square$

From the above we have:

**Lemma 1.4** *Let  $\mathbb{K}$  be a variety of MV-algebras.  $\mathbb{K}$  is a locally finite variety if and only if  $\mathbb{K} = \mathbb{V}(L_{n_1}, \dots, L_{n_r})$  for some  $n_1, \dots, n_r \in \omega$*

**Proof :** By Theorem 1.3, for every  $r < \omega$  and any  $n_1, \dots, n_r \in \omega$ ,  $\mathbb{V}(L_{n_1}, \dots, L_{n_r})$  is a locally finite variety.

Since  $L_1^\omega \in \mathbb{W}$  is infinite and it is finitely generated by the element  $(0, 1)$ ,  $\mathbb{W}$  is not locally finite.

Assume  $\mathbb{K}$  is a proper variety of  $\mathbb{W}$ . If  $\mathbb{K}$  is not of the form  $\mathbb{V}(L_{n_1}, \dots, L_{n_r})$  for some  $n_1, \dots, n_r \in \omega$ , then by Theorem 1.2 we have that  $\mathbb{K} = \mathbb{V}(L_i \mid i \in I, L_j^\omega \mid j \in J)$  with  $J \neq \emptyset$ . Hence there is  $j \in J$  such that  $L_j^\omega \in \mathbb{K}$ . Since  $L_j^\omega$  is infinite and it is finitely generated by  $(0, 1), (\frac{1}{j}, 0) \in L_j^\omega$ ,  $L_j^\omega$  is not locally finite and therefore,  $\mathbb{K}$  is not a locally finite variety.  $\square$

## 2 Locally finite quasivarieties and critical algebras.

We want to obtain all locally finite quasivarieties of MV-algebras. First we observe that from Theorem 1.3 it follows.

**Corollary 2.1** *Every finitely generated quasivariety is locally finite.*  $\square$

And from Lemma 1.4 we obtain:

**Corollary 2.2** *A quasivariety of MV-algebras is locally finite if and only if it is a subquasivariety of a variety of the form  $\mathbb{V}(L_{n_1}, \dots, L_{n_r})$  for some  $n_1, \dots, n_r \in \omega$ .*  $\square$

A **critical** algebra is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras. The interest of critical algebras is given by the following result, which is mentioned in [10, page 128], but no proof is given.

**Theorem 2.3** *Every locally finite quasivariety is generated by its critical algebras.*

**Proof :** Let  $\mathbb{K}$  be a locally finite quasivariety and  $\{\mathbf{A}_i : i \in I\} \subseteq \mathbb{K}$  the family of all critical algebras contained in  $\mathbb{K}$ . Obviously,  $\mathbb{Q}(\{\mathbf{A}_i : i \in I\}) \subseteq \mathbb{K}$ . Assume  $\bigwedge_{k=1}^m \varphi_k \approx \psi_k \Rightarrow \varphi \approx \psi(x_0, \dots, x_n)$  is a quasiequation not satisfied by  $\mathbb{K}$ . Therefore, there exist an algebra  $\mathbf{A} \in \mathbb{K}$  and  $a_0, \dots, a_n \in \mathbf{A}$  such that

$$\mathbf{A} \not\models \bigwedge_{k=1}^m \varphi_k \approx \psi_k \Rightarrow \varphi \approx \psi(a_0, \dots, a_n)$$

Let  $\llbracket a_0, \dots, a_n \rrbracket_{\mathbf{A}}$  be the subalgebra of  $\mathbf{A}$  generated by  $\{a_0, \dots, a_n\}$ . Since  $\mathbb{K}$  is locally finite,

a)  $\llbracket a_0 \dots, a_n \rrbracket_{\mathbf{A}}$  is finite

b)  $\llbracket a_0 \dots, a_n \rrbracket_{\mathbf{A}} \not\models \bigwedge_{k=1}^m \varphi_k \approx \psi_k \Rightarrow \varphi \approx \psi$

It is easy to see that a quasivariety generated by a finite algebra  $\mathbf{B}$  is generated by all critical subalgebras of  $\mathbf{B}$  (The proof is straightforward by induction over the cardinal of  $\mathbf{B}$ ). Therefore, from a) and b) we deduce that there is a critical algebra  $\mathbf{A}_l \subseteq \llbracket a_0 \dots, a_n \rrbracket_{\mathbf{A}}$  such that

$$\mathbf{A}_l \not\models \bigwedge_{k=1}^m \varphi_k \approx \psi_k \Rightarrow \varphi \approx \psi$$

So, since  $\mathbf{A}_l \in \{\mathbf{A}_i : i \in I\}$  and a quasivariety is a class of algebras definable by means of quasiequations,  $\mathbb{K} \subseteq \mathbb{Q}(\{\mathbf{A}_i : i \in I\})$   $\square$

Our next purpose is to characterize critical MV-algebras. We need a previous result.

**Lemma 2.4** *If  $L_{n_0} \times \dots \times L_{n_{l-1}}$  is embeddable into  $\prod_{j \in J} L_{m_j}$  where the set  $\{m_j : j \in J\}$  is finite then*

1. *For every  $i < l$  there exists  $j \in J$  such that  $n_i | m_j$ .*
2. *For every  $j \in J$  there exists  $i < l$  such that  $n_i | m_j$ .*

**Proof :** 1) If  $L_{n_0} \times \dots \times L_{n_{l-1}}$  is embeddable into  $\prod_{j \in J} L_{m_j}$ , then

$$L_{n_1} \times \dots \times L_{n_l} \in \mathbb{V}(\prod_{j \in J} L_{m_j}) = \mathbb{V}(\{L_{m_j} ; j \in J\}).$$

Hence, for every  $i < l$ ,  $L_{n_i} \in \mathbb{V}(\{L_{m_j} ; j \in J\})$ . Since  $\{m_j : j \in J\}$  is finite, from a result due to Jónsson [3, page 149], we deduce that the class subdirectly irreducible members of  $\mathbb{V}(\{L_{m_j} ; j \in J\})$  is  $\mathbb{I}(\{L_n : \exists j \in J \ L_n \subseteq L_{m_j}\})$ . Since  $L_{n_i}$  is simple, therefore subdirectly irreducible, for every  $i < l$  there exists  $j \in J$  such that  $L_{n_i} \subseteq L_{m_j}$ , and by 4 of Theorem 1.1  $n_i | m_j$ .

2) For each  $j \in J$  consider the natural projection:  $\pi_j : \prod_{j \in J} L_{m_j} \longrightarrow L_{m_j}$ . Let

$\gamma : L_{n_0} \times \dots \times L_{n_{l-1}} \rightarrow \prod_{j \in J} L_{m_j}$  be an embedding, then for every  $j \in J$   $\gamma_j = \pi_j \circ \gamma$  is an homomorphism from  $L_{n_0} \times \dots \times L_{n_{l-1}}$  to  $L_{m_j}$ . Hence

$$L_{n_0} \times \dots \times L_{n_{l-1}} / \text{Ker}(\gamma_j) \cong \gamma_j(L_{n_0} \times \dots \times L_{n_{l-1}}) \subseteq L_{m_j}$$

So,  $L_{n_0} \times \dots \times L_{n_{l-1}} / \text{Ker}(\gamma_j)$  is simple, and by [6, Theorem 4.1.19] we have that  $\text{Ker}(\gamma_j)$  is a maximal congruence relation of  $L_{n_0} \times \dots \times L_{n_{l-1}}$ . From [7, Lemma 2.3]

(see also [16]) and the fact that all  $\mathbf{L}_n$ 's are simple, it can be deduced that there is  $k < l$  such that

$$\text{Ker}(\gamma_j) = \mathbf{L}_{n_0}^2 \times \cdots \times \mathbf{L}_{n_{k-1}}^2 \times \Delta_{\mathbf{L}_{n_k}} \times \mathbf{L}_{n_{k+1}}^2 \times \cdots \times \mathbf{L}_{n_{l-1}}^2.$$

Hence, for every  $j \in J$  there exists  $k < l$  such that

$$\mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{l-1}} / \text{Ker}(\gamma_j) \cong \mathbf{L}_{n_k} \subseteq \mathbf{L}_{m_j}.$$

Thus  $n_k | m_j$ . □

Finally we give a characterization of all critical MV-algebras.

**Theorem 2.5** *An MV-algebra  $\mathbf{A}$  is critical if and only if  $\mathbf{A}$  is isomorphic to a finite MV-algebra  $\mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{l-1}}$  satisfying the following conditions:*

1. *For every  $i, j < l$ ,  $i \neq j$  implies  $n_i \neq n_j$ .*
2. *If there exists  $n_j$   $j < l$  such that  $n_i | n_j$  for some  $i \neq j$ , then  $n_j$  is unique.*

**Proof :** Assume that  $\mathbf{A} = \mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{l-1}}$  satisfies conditions 1) and 2). First, we will show the following:

**Claim:** *Every proper subalgebra of  $\mathbf{A}$  is embeddable into a subalgebra of  $\mathbf{A}$  of the form  $\mathbf{L}_{d_0} \times \cdots \times \mathbf{L}_{d_{l-1}}$ , where  $d_i | n_i$  for each  $i < l$  and there exists  $j < l$  such that  $d_j \neq n_j$ .*

**Proof of the claim:** Let  $\mathbf{B}$  be a proper subalgebra of  $\mathbf{A}$ . Since  $\mathbf{A}$  is finite  $\mathbf{B}$  is also finite and by Theorem 1.1,  $\mathbf{B}$  is isomorphic to  $\mathbf{L}_{p_0} \times \cdots \times \mathbf{L}_{p_{r-1}}$ . For each  $i < l$  consider the natural projection:  $\pi_i : \mathbf{A} \rightarrow \mathbf{L}_{n_i}$ , if for all  $i < l$ , we write  $\gamma_i = \pi_i \upharpoonright_{\mathbf{B}}$ , then we can assume that  $\mathbf{B}$  is embeddable into  $\gamma_0(\mathbf{B}) \times \cdots \times \gamma_{l-1}(\mathbf{B})$ . Moreover, since  $\gamma_i(\mathbf{B}) \subseteq \mathbf{L}_{n_i}$ , we have  $\gamma_i(\mathbf{B}) = \mathbf{L}_{d_i}$  for some  $d_i | n_i$ .

Assume that  $\gamma_i(\mathbf{B}) = \mathbf{L}_{n_i}$  for each  $i < l$ . Then, for every  $i < l$ ,  $\mathbf{B} / \text{Ker}(\gamma_i) \cong \mathbf{L}_{n_i}$  and since  $\mathbf{L}_{n_i}$  is simple,  $\text{Ker}(\gamma_i)$  is a maximal congruence relation of  $\mathbf{B}$ . From [7, Lemma 2.3] (see also [16]) and the fact that all  $\mathbf{L}_n$ 's are simple, it can be deduced that there is  $k < r$  such that

$$\text{Ker}(\gamma_i) = \mathbf{L}_{p_0}^2 \times \cdots \times \mathbf{L}_{p_{k-1}}^2 \times \Delta_{\mathbf{L}_{p_k}} \times \mathbf{L}_{p_{k+1}}^2 \times \cdots \times \mathbf{L}_{p_{r-1}}^2.$$

Hence, for every  $i < l$  there exists  $k < r$  such that  $\mathbf{L}_{p_k} = \mathbf{L}_{n_i}$ . By condition (1),  $i \neq j$  implies  $n_i \neq n_j$ , so  $l \leq r$  and

$$\mathbf{B} \cong \mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{l-1}} \times \mathbf{L}_{m_l} \times \cdots \times \mathbf{L}_{m_{r-1}} = \mathbf{A} \times \mathbf{L}_{m_l} \times \cdots \times \mathbf{L}_{m_{r-1}}$$

that implies  $|\mathbf{A}| \leq |\mathbf{B}|$ , which contradicts that  $\mathbf{B}$  is a proper subalgebra of  $\mathbf{A}$ . And the claim is proved.

Suppose that  $\mathbf{A} \in \mathbb{Q}(\{\mathbf{B} \subsetneq \mathbf{A}\})$ , then

$$\mathbf{A} \in \text{ISPP}_U(\{\mathbf{L}_{d_0} \times \cdots \times \mathbf{L}_{d_{l-1}} : \forall i \, d_i | n_i; \exists k \, d_k \neq n_k\}).$$

Since  $\{\mathbb{L}_{d_0} \times \cdots \times \mathbb{L}_{d_{l-1}} : \forall i d_i | n_i; \exists k d_k \neq n_k\}$  is a finite set of finite MV-algebras, we have that  $\mathbf{A} \in \mathbb{ISP}(\{\mathbb{L}_{d_0} \times \cdots \times \mathbb{L}_{d_{l-1}} : \forall i d_i | n_i; \exists k d_k \neq n_k\})$ . Thus  $\mathbb{L}_{n_0} \times \cdots \times \mathbb{L}_{n_{l-1}}$  is embeddable into  $\prod_{k < n} (\mathbb{L}_{d_{0,k}} \times \cdots \times \mathbb{L}_{d_{l-1,k}})^{\alpha_k}$  where

$$\{\mathbb{L}_{d_{0,k}} \times \cdots \times \mathbb{L}_{d_{l-1,k}} : k < n\} \subseteq \{\mathbb{L}_{d_0} \times \cdots \times \mathbb{L}_{d_{l-1}} : \forall i d_i | n_i; \exists k d_k \neq n_k\}.$$

Since the set  $\{d_{t,k} : t < l; k \leq n\}$  is finite, we can apply Lemma 2.4. If there exists  $i, j < l$  such that  $i \neq j$  and  $n_i | n_j$ , then by condition 2),  $n_j$  is unique. By Lemma 2.4, there exists  $\mathbb{L}_{d_{t,m}}$  such that  $n_j | d_{t,m}$ . That is, there exists

$$\mathbb{L}_{d_{0,m}} \times \cdots \times \mathbb{L}_{d_{l-1,m}} \in \{\mathbb{L}_{d_0} \times \cdots \times \mathbb{L}_{d_{l-1}} : \forall i d_i | n_i; \exists k d_k \neq n_k\}$$

such that  $n_j | d_{t,m}$  for some  $t < l$ . By condition 2),  $n_j$  does not divide any  $n_i$  other than itself. Therefore, since  $d_{t,m}$  is a divisor of  $n_t$ , we have that  $n_t = d_{t,m} = n_j$  and by 1),  $t = j$ . Since

$$\mathbb{L}_{d_{0,m}} \times \cdots \times \mathbb{L}_{d_{l-1,m}} \in \{\mathbb{L}_{d_0} \times \cdots \times \mathbb{L}_{d_{l-1}} : \forall i d_i | n_i; \exists k d_k \neq n_k\},$$

there exists  $r \neq j < l$  such that  $d_{r,m} | n_r$  and  $d_{r,m} \neq n_r$ . By 2) of Lemma 2.4, there exists  $s, r < l$  such that  $s \neq r < l$  and  $n_s | d_{r,m}$ . Thus  $n_s | n_r$ ,  $n_i | n_j$ ,  $r \neq s$ ,  $i \neq j$  and  $r \neq j$ , which contradicts condition 2).

If for all  $1 \leq i, j \leq l$  such that  $i \neq j$ ,  $n_i \nmid n_j$ , then the same argument follows by taking any  $n_j$ ,  $j < l$ .

Since  $\mathbf{A}$  is finite and  $\mathbf{A} \notin \mathbb{Q}(\{\mathbf{B} \subsetneq \mathbf{A}\})$ ,  $\mathbf{A}$  is critical.

Conversely, if  $\mathbf{A}$  is a critical MV-algebra, then  $\mathbf{A}$  is finite and by Theorem 1.1, we can suppose, without loss of generality, that

$$\mathbf{A} = \mathbb{L}_{n_0}^{m_0} \times \cdots \times \mathbb{L}_{n_{k-1}}^{m_{k-1}}$$

for some  $n_0, \dots, n_{k-1}, m_0, \dots, m_{k-1} \in \omega$  and  $n_i \neq n_j$  when  $i \neq j$ . If not all  $m_i$ 's are equal to 1, then the correspondence

$$\alpha: (a(0), \dots, a(k-1)) \mapsto \alpha(a) = \left( \overbrace{(a(0), \dots, a(0))}^{m_0}, \dots, \overbrace{(a(k-1), \dots, a(k-1))}^{m_{k-1}} \right)$$

defines an isomorphism from  $\mathbb{L}_{n_0} \times \cdots \times \mathbb{L}_{n_{k-1}}$  onto a proper subalgebra of  $\mathbf{A}$ . Let  $m = \max\{m_0, \dots, m_{k-1}\}$ , then the correspondence

$$\beta: \mathbb{L}_{n_0}^{m_0} \times \cdots \times \mathbb{L}_{n_{k-1}}^{m_{k-1}} \rightarrow \mathbb{L}_{n_0}^m \times \cdots \times \mathbb{L}_{n_{k-1}}^m$$

such that for every  $r < k$ ,

$$\beta((b(0), \dots, b(k-1)))(r) = (b(r)(1), \dots, b(r)(m_r), \overbrace{b(r)(1), \dots, b(r)(1)}^{m-m_r}),$$

gives an embedding from  $\mathbf{A}$  into  $\mathbf{L}_{n_0}^m \times \cdots \times \mathbf{L}_{n_{k-1}}^m \cong (\mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{k-1}})^m$ . Thus  $\mathbf{A} \in \mathbb{Q}(\mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{k-1}})$ . Since  $\mathbf{A}$  is critical, we have  $m_0, \dots, m_{k-1} = 1$ . Hence it satisfies condition 1). Suppose condition 2) fails, then there exist  $i \neq j$  and  $s \neq r$  such that  $n_i | n_j$ ,  $n_s | n_r$  and  $j \neq r$ . Since  $n_i | n_j$ , we have that the correspondence that maps

$$(a(0), \dots, a(j-1), a(j+1), \dots, a(k-1))$$

to

$$(a(0), \dots, a(j-1), a(i), a(j+1), \dots, a(k-1)).$$

defines an isomorphism from  $\mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{j-1}} \times \mathbf{L}_{n_{j+1}} \times \cdots \times \mathbf{L}_{n_{k-1}}$  onto a proper subalgebra of  $\mathbf{A}$ .

Similarly the algebra  $\mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{r-1}} \times \mathbf{L}_{n_{r+1}} \times \cdots \times \mathbf{L}_{n_{k-1}}$  is isomorphic to a proper subalgebra of  $\mathbf{A}$ . Finally, observe that  $\mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{k-1}}$  is embeddable into

$$\mathbf{L}_{n_1}^2 \times \cdots \times \mathbf{L}_{n_{j-1}}^2 \times \mathbf{L}_{n_j} \times \mathbf{L}_{n_{j+1}}^2 \times \cdots \times \mathbf{L}_{n_{r-1}}^2 \times \mathbf{L}_{n_r} \times \mathbf{L}_{n_{r+1}}^2 \times \cdots \times \mathbf{L}_{n_k}^2,$$

by means of the correspondence  $\delta$  defined as:

$$\delta(a(0), \dots, a(k-1))(i) = \begin{cases} (a(i), a(i)), & \text{if } i \neq j, r; \\ a(i), & \text{if } i = j, r. \end{cases}$$

Therefore

$\mathbf{A} \in \mathbb{Q}(\mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{j-1}} \times \mathbf{L}_{n_{j+1}} \times \cdots \times \mathbf{L}_{n_{k-1}}, \mathbf{L}_{n_0} \times \cdots \times \mathbf{L}_{n_{r-1}} \times \mathbf{L}_{n_{r+1}} \times \cdots \times \mathbf{L}_{n_{k-1}})$  in contradiction with the fact that  $\mathbf{A}$  is critical.  $\square$

**Corollary 2.6** *The number of non isomorphic critical MV-algebras in a proper variety of  $\mathbb{W}$  is finite.*  $\square$

**Proof :** If  $\mathbb{K}$  is a proper variety of  $\mathbb{W}$ , then it is shown in [13] and [6] that  $\mathbb{K}$  contains a finite number of  $\mathbf{L}_n$ 's. Let  $M = \{n \in \omega : \mathbf{L}_n \in \mathbb{K}\}$ , clearly  $|M| < \omega$ . By Theorem 2.5, all critical algebras in  $\mathbb{K}$  are:

$$\mathbb{I}(\{\mathbf{L}_{n_1} \times \cdots \times \mathbf{L}_{n_l} : \text{satisfying (1) and (2) of Theorem 2.5 and } n_i \in M\}).$$

Since  $|M|$  is finite we have that the number of non isomorphic critical MV-algebras in  $\mathbb{K}$  is finite.  $\square$

From the above result we deduce :

**Theorem 2.7** *A quasivariety of MV-algebras is locally finite if and only if it is finitely generated.*

**Proof :** Let  $\mathbb{K}$  be a locally finite quasivariety of MV-algebras, by Corollary 2.2,  $\mathbb{V}(\mathbb{K})$  is a proper subvariety of  $\mathbb{W}$ , thus, applying Corollary 2.6 the number of non isomorphic critical MV-algebras in  $\mathbb{K}$  is finite. By Theorem 2.3,  $\mathbb{K}$  is generated by its critical algebras, therefore, since a quasivariety is closed under the operation of isomorphic images,  $\mathbb{K}$  is finitely generated.

The converse is given by Corollary 2.1  $\square$



**Corollary 2.8** *Every locally finite variety of MV-algebras contains a finite number of quasivarieties and, therefore, the class of locally finite quasivarieties of MV-algebras is countable.*  $\square$

In order to classify and distinguish locally finite quasivarieties by looking at their generators we need the following result:

**Lemma 2.9** *Let  $\{L_{n_{i1}} \times \cdots \times L_{n_{il(i)}} : i \in I\}$  and  $\{L_{m_{j1}} \times \cdots \times L_{m_{jl(j)}} : j \in J\}$  two finite families of critical MV-algebras, then*

$$\mathbb{Q}(\{L_{n_{i1}} \times \cdots \times L_{n_{il(i)}} : i \in I\}) \subseteq \mathbb{Q}(\{L_{m_{j1}} \times \cdots \times L_{m_{jl(j)}} : j \in J\})$$

*if and only if*

*For every  $i \in I$ , there exists  $H$ , a non-empty subset of  $J$ , such that:*

1. *For any  $1 \leq k \leq l(i)$  there are  $j \in H$  and  $1 \leq r \leq l(j)$  such that  $n_{ik} | m_{jr}$ .*
2. *For any  $j \in H$  and  $1 \leq r \leq l(j)$  there exists  $1 \leq k \leq l(i)$  such that  $n_{ik} | m_{jr}$ .*

**Proof :** Assume  $\mathbb{Q}(\{L_{n_{i1}} \times \cdots \times L_{n_{il(i)}} : i \in I\}) \subseteq \mathbb{Q}(\{L_{m_{j1}} \times \cdots \times L_{m_{jl(j)}} : j \in J\})$ , then for every  $i \in I$ , there is  $\emptyset \neq H \subseteq J$  such that  $L_{n_{i1}} \times \cdots \times L_{n_{il(i)}}$  is embeddable into  $\prod_{j \in H} (L_{m_{j1}} \times \cdots \times L_{m_{jl(j)}})^{\alpha_j}$ . Therefore, since  $\bigcup_{j \in H} \{m_{jr} : r \leq l(j)\}$  is a finite set then, it follows from lemma 2.4 conditions 1 and 2 .

To prove the converse we show that for every  $i \in I$ ,

$$L_{n_{i1}} \times \cdots \times L_{n_{il(i)}} \in \mathbb{ISP}(\{L_{m_{j1}} \times \cdots \times L_{m_{jl(j)}} : j \in H\})$$

where  $H$  is the subset of  $J$  defined in the hypothesis By condition 1), for every  $1 \leq k \leq l(i)$ , we choose  $j \in H$  named  $j_k$  and we choose  $1 \leq r_k \leq l(j_k)$  such that  $n_k | m_{j_k r_k}$ , the the following map

$$\beta : L_{n_{i1}} \times \cdots \times L_{n_{il(i)}} \rightarrow \prod_{1 \leq k \leq l(i)} L_{m_{j_k 1}} \times \cdots \times L_{m_{j_k l(j_k)}} : \bar{a} \mapsto \beta(\bar{a})$$

$$\text{on } \beta(\bar{a})(k)(r) = \begin{cases} \bar{a}(k) & \text{if } r = r_k \\ \bar{a}(l) & \text{for some } 1 \leq l \leq l(i) \text{ such that } n_l | m_{j_k r} \text{ if } r \neq r_k \\ & \text{it exists by condition 2)} \end{cases}$$

gives an embedding.  $\square$

It is well known that every subvariety of MV-algebras is finitely axiomatizable. In fact, some effective axiomatizations are given in [9] and in [15]. To our concern, we only need to axiomatize locally finite varieties of MV-algebras. In [18] it is proved that the variety generated by  $L_n$  is finitely axiomatizable and it is axiomatized by

MV1,...,MV6 plus a single axiom of the form  $\varphi(x) \approx 1$ , denoted by  $v_n(x) \approx 1$ . Moreover, for every  $n_1, \dots, n_r < \omega$ ,  $\mathbb{V}(L_{n_1}, \dots, L_{n_r})$  is the subvariety of  $\mathbb{W}$  defined by the equation  $v_{n_1}(x) \vee \dots \vee v_{n_r}(x) \approx 1$  [18, Theorem 1.8]. Where  $\vee$  is defined by  $x \vee y = \neg(\neg x \oplus y) \oplus y$ .

In general, locally finite quasivarieties are not finitely axiomatizable, not even finitely generated quasivarieties are finitely axiomatizable. For instance: Let  $\mathbb{K} = \mathbb{Q}(\mathbf{A})$  where  $\mathbf{A} = \langle \{0, 1, 2\}, f, g \rangle$  is of type (1,1) with  $f$  and  $g$  defined by  $f(0) = 1, g(0) = 2$  and  $f(x) = g(x) = x$  for  $x \neq 0$ . Due to Gorbunov [12],  $\mathbb{K}$  is not finitely axiomatizable while it is finitely generated.(see also [8, page 149]). In the case of MV-algebras, we will show that locally finite quasivarieties of MV-algebras are finitely axiomatizable.

**Theorem 2.10** *Every locally finite quasivariety of MV-algebras is finitely axiomatizable.*

**Proof :** Let  $\mathbb{K}$  be a locally finite quasivariety of MV-algebras, by Corollary 2.2,  $\mathbb{V}(\mathbb{K})$  is a proper locally finite subvariety of  $\mathbb{W}$  and therefore it is finitely axiomatizable. Since  $\mathbb{K}$  is finitely generated and  $\mathbb{V}(\mathbb{K})$  is finitely axiomatizable we only need to prove that  $\mathbb{K}$  can be axiomatized by a set of quasiequations with a finite number of variables (see [1, Lemma 2.3]).

Let  $\Sigma_n$  be the set of quasiequations with at most  $n$  variables satisfied by  $\mathbb{K}$  and  $Mod(\Sigma_n)$  be the family of algebras of same type satisfying  $\Sigma_n$ . Observe that, since every locally finite variety of MV-algebras is finitely axiomatizable with two variables,  $Mod(\Sigma_n)$  is a quasivariety contained in  $\mathbb{V}(\mathbb{K})$  for every  $n \geq 2$ . Moreover,  $Mod(\Sigma_n) \supseteq Mod(\Sigma_m) \supseteq \mathbb{K}$  for every  $n < m$ . Assume that  $\mathbb{K}$  is not axiomatizable with a finite number variables, then there is an infinite sequence  $n_1, n_2, n_3, \dots$  such that  $\mathbb{V}(\mathbb{K}) \supseteq Mod(\Sigma_{n_1}) \supsetneq Mod(\Sigma_{n_2}) \supsetneq Mod(\Sigma_{n_3}) \supsetneq \dots \supsetneq \mathbb{K}$  which contradicts the fact that  $\mathbb{V}(\mathbb{K})$  contains a finite number of quasivarieties of MV-algebras (Corollary 2.8).  $\square$

### 3 Two examples

In order to show the difficulty and/or simplicity to obtain axiomatizations of locally finite quasivarieties of MV-algebras we give two examples:

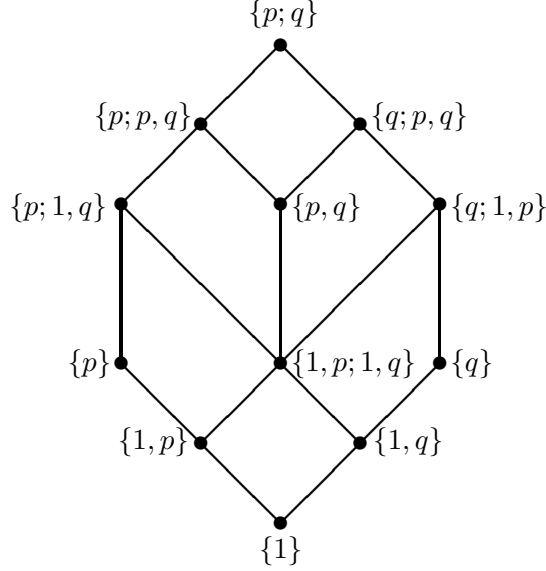
**Example 3.1** *Quasivarieties contained in  $\mathbb{V}(L_p, L_q)$ , where  $p$  and  $q$  are two distinct prime natural numbers.*

Since the only divisors of  $p$  and  $q$  are 1,  $p, q$ , by the characterization given in Theorem 2.5, we have that all critical MV-algebras contained in  $\mathbb{V}(L_p, L_q)$  are

$$\mathbb{I}(\{L_1, L_p, L_q, L_1 \times L_p, L_1 \times L_q, L_p \times L_q\}).$$

Applying Lemma 2.9 we obtain all subquasivarieties of  $\mathbb{V}(\mathbf{L}_p, \mathbf{L}_q)$  which are sketched in figure 1.

Figure 1: Lattice of all quasivarieties contained in  $\mathbb{V}(\mathbf{L}_p, \mathbf{L}_q)$ .



$\{n_1, \dots, n_l; m_1, \dots, m_k\}$  stands for  $\mathbb{Q}(\mathbf{L}_{n_1} \times \dots \times \mathbf{L}_{n_l}, \mathbf{L}_{m_1} \times \dots \times \mathbf{L}_{m_k})$ .

Given a quasivariety  $\mathbb{K}$  of MV-algebras, let

$$\mathbb{K} : \mathbf{L}_n = \{\mathbf{A} \in \mathbb{K} : \mathbf{L}_n \notin \mathbb{IS}(\mathbf{A})\}.$$

We define  $0x = 0$  and for each  $n \in \omega$   $(n+1)x = x \oplus nx$ . It is known (see [18, Lemma 2.2.]) that  $\mathbf{L}_n$  is embeddable into an MV-algebra  $\mathbf{A}$  if, and only if, there is an element  $a \in \mathbf{A}$  such that  $(n-1)(\neg a) = a$  and  $a \neq 1^{\mathbf{A}}$ . Thus the quasiequation  $(n-1)(\neg x) \approx x \Rightarrow x \approx 1$  holds in an MV-algebra  $\mathbf{A}$  if and only if  $\mathbf{A}$  does not contain a copy of  $\mathbf{L}_n$ . Therefore,  $\mathbb{K} : \mathbf{L}_n$  is the quasivariety axiomatized by:

$$\{\text{axioms of } \mathbb{K}\} \cup \{(n-1)(\neg x) \approx x \Rightarrow x \approx 1\}.$$

From the above, with more or less difficulty, one can prove the following:

1.  $\mathbb{Q}(\{\mathbf{L}_p, \mathbf{L}_p \times \mathbf{L}_q\}) = \mathbb{V}(\{\mathbf{L}_p, \mathbf{L}_q\}) : \mathbf{L}_q$ .
2.  $\mathbb{Q}(\{\mathbf{L}_q, \mathbf{L}_p \times \mathbf{L}_q\}) = \mathbb{V}(\{\mathbf{L}_p, \mathbf{L}_q\}) : \mathbf{L}_p$ .
3.  $\mathbb{Q}(\mathbf{L}_p \times \mathbf{L}_q) = (\mathbb{V}(\{\mathbf{L}_p, \mathbf{L}_q\}) : \mathbf{L}_p) : \mathbf{L}_q$ .

4.  $\mathbb{Q}(\mathbb{L}_1 \times \mathbb{L}_p) = \mathbb{V}(\mathbb{L}_p) : \mathbb{L}_p.$

5.  $\mathbb{Q}(\mathbb{L}_1 \times \mathbb{L}_q) = \mathbb{V}(\mathbb{L}_q) : \mathbb{L}_q.$

6.  $\mathbb{Q}(\mathbb{L}_p, \mathbb{L}_1 \times \mathbb{L}_q)$  is axiomatized by the axioms of  $\mathbb{V}(\mathbb{L}_p, \mathbb{L}_q) : \mathbb{L}_q$  and

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow v_p(y) \approx 1, \text{ where } r = \max\{p, q\}.$$

7.  $\mathbb{Q}(\mathbb{L}_q, \mathbb{L}_1 \times \mathbb{L}_p)$  is axiomatized by the axioms of  $\mathbb{V}(\mathbb{L}_p, \mathbb{L}_q) : \mathbb{L}_p$  and

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow v_q(y) \approx 1, \text{ where } r = \max\{p, q\}.$$

8.  $\mathbb{Q}(\mathbb{L}_1 \times \mathbb{L}_p, \mathbb{L}_1 \times \mathbb{L}_q)$  is axiomatized by the axioms of  $(\mathbb{V}(\mathbb{L}_p, \mathbb{L}_q) : \mathbb{L}_q) : \mathbb{L}_p$  and

$$rx \approx 1 \ \& \ r(\neg x) \approx 1 \Rightarrow x \approx 1, \text{ where } r = \max\{p, q\}.$$

**Example 3.2** *Quasivarieties contained in  $\mathbb{V}(\mathbb{L}_{p^r})$ , where  $p$  is a prime natural number and  $r$  is a natural number.*

The class of its critical algebras is given by:

$$\mathbb{I}\left(\{\mathbb{L}_{p^s} : s \leq r\} \cup \{\mathbb{L}_{p^n} \times \mathbb{L}_{p^m} : n < m \leq r\}\right),$$

After lengthy computations (some cases are not straightforward), one can prove that each subquasivariety of  $\mathbb{V}(\mathbb{L}_{p^r})$  belongs to one of the following three types:

1.  $\mathbb{Q}(\mathbb{L}_{p^s}) = \mathbb{V}(\mathbb{L}_{p^s})$  where  $s \leq r$  and it is axiomatized by

(a) MV1,...,MV6 and

(b)  $v_{p^s}(x) \approx 1.$

2.  $\mathbb{Q}(\mathbb{L}_{p^{n_1}} \times \mathbb{L}_{p^{m_1}}, \dots, \mathbb{L}_{p^{n_k}} \times \mathbb{L}_{p^{m_k}})$  such that  $n_i < m_i \leq r$ , for every  $1 \leq i \leq k$  and  $n_i < n_j$  and  $m_i > m_j$  if  $i < j$  and it is axiomatized by:

(a) MV1,...,MV6,

(b)  $v_{p^{m_1}}(x) \approx 1,$

(c)  $(p^{n_k+1} - 1)(\neg x) \approx x \Rightarrow x \approx 1$  and

(d)  $(p^{n_{j-1}+1} - 1)(\neg x) \approx x \Rightarrow v_{p^{m_j}}(y) \approx 1$  for every  $2 \leq j \leq k.$

3.  $\mathbb{Q}(\mathbb{L}_{p^{n_1}} \times \mathbb{L}_{p^{m_1}}, \dots, \mathbb{L}_{p^{n_k}} \times \mathbb{L}_{p^{m_k}}, \mathbb{L}_{p^s})$  such that  $n_i < s < m_i \leq r$ , for every  $1 \leq i \leq k$  and  $n_i < n_j$  and  $m_i > m_j$  if  $i < j$  and it is axiomatized by:

(a) MV1,...,MV6,

(b)  $v_{p^{m_1}}(x) \approx 1,$

- (c)  $(p^{s+1} - 1)(\neg x) \approx x \Rightarrow x \approx 1$ ,
- (d)  $(p^{n_{j-1}+1} - 1)(\neg x) \approx x \Rightarrow v_{p^{m_j}}(y) \approx 1$  for every  $2 \leq j \leq k$  and
- (e)  $(p^{n_k+1} - 1)(\neg x) \approx x \Rightarrow v_{p^s}(y) \approx 1$ .

It remains an *open problem* whether we can find a computational method to obtain for every locally finite quasivariety its axiomatization.

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